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A FULL-STRENGTH ORIFICE UNDER CONDITIONS OF GEOMETRIC NONLINEARITY

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In studies of the equilibrium of a deformable solid body using any model of elasticity, both direct and inverse problems are of interest. In inverse problems, the quantities determined in direct problems are given beforehand, and the quantities that are usually given are to be determined. The problem of a full-strength orifice also belongs to the group of inverse problems of elasticity.

A construction with an orifice begins to disintegrate or lose its supporting capacity at the sites of the highest stress concentration on the contour of the orifice. This does not proceed over the entire contour at once, but first at certain points, and this determines the admissible level of loads. A full-strength orifice is distinguished by equal stress concentration over the entire contour. Such a contour preserves or loses strength simultaneously at all points. A full-strength orifice usually collapses at a higher level of loads, and this determines the effectiveness of these constructions.

Full-strength orifices were found within the framework of linear elasticity theory and plasticity theory in a number of works (see, e.g., [1, 2]). Below, this problem is studied in V. V. Novozhilov's variant of geometrically nonlinear elasticity [3].

We consider an infinite plate S which is weakened by an orifice with smooth contour L. It is assumed that volume forces are absent, the stresses and rotation at infinity are given, and the orifice contour is loaded by normal and tangential stresses of constant intensity. It is required to determine the stress and rotation fields and an orifice shape such that the stresses on the surfaces normal to the contour would also be constant quantities. Thus, the conditions at infinity and on the contour have the form

$$P_{xx} = P_{xx}^{\infty}, \quad P_{yy} = P_{yy}^{\infty}, \quad P_{xy} = P_{xy}^{\infty}, \quad \omega_{xy} = \omega_{xy}^{\infty} \quad \text{at} \quad \infty; \tag{1}$$

$$P_{nn} = p = \text{const}, \quad P_{tt} = \sigma = \text{const}, \quad P_{nt} = \tau = \text{const} \quad \text{at} \quad L, \tag{2}$$

where P_{xx}, P_{yy}, P_{xy}, and ω_{xy} are the stress and rotation components on the Cartesian axes x and y; P_{nn}, P_{tt}, and P_{nt} are the stress components on the natural axes of the contour [the normal n and the tangent t (Fig. 1)]; p and τ are given constants; and σ is to be determined. These conditions formulate the problem of a full-strength orifice.

We shall solve the formulated problem using the geometrically nonlinear elasticity variant [3]. This variant assumes that the elongation-shears of material elements are on the same order as the squares of their rotation; because of this, the strain components are functions that are linear with respect to the former quantities and square with respect the latter. It is also assumed that the mechanical behavior of the material is described by Hooke's law.

The assumptions of this model are usually realized in flexible bodies, and in bodies with orifices near their internal and external boundaries. Precisely the latter circumstance is responsible for the use of Novozhilov's model for the solution of the formulated problem.

As was established in [4], in this model, stresses and rotation under planar deformation can be expressed in terms of complex potentials $\varphi(z)$ and $\psi(z)$ by nonlinear formulas which generalize Kolosov's formulas of

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linear elasticity [5] and, in the absence of volume forces, have the form

$$P^{11} = \overline{P^{22}} = -2\left(z\,\overline{\varphi''(z)} + \overline{\psi'(z)}\right) - 2k\,\overline{\varphi''(z)}\left(z\,\overline{\varphi'(z)} - \varphi(z)\right),$$
$$P^{21} = P^{12} = 2\left(\varphi'(z) + \overline{\varphi'(z)}\right) + k\left(\varphi'(z) - \overline{\varphi'(z)}\right)^2, \tag{3}$$
$$\omega^{21} = \overline{\omega^{12}} = 2k\left(\varphi'(z) - \overline{\varphi'(z)}\right), \quad k = (1-\nu)/\mu.$$

Here $z = x + iy$ and $\bar z = x - iy$ are complex variables; μ is the modulus of shear; ν is Poisson's ratio; the bar above a quantity indicates complex conjugation; the prime denotes the derivative of the function; and $P^{\alpha\beta}$ and $\omega^{\alpha\beta}$ are the complex components of stresses and rotation related to the Cartesian components of the same quantities by the expressions

$$P^{11} = \overline{P^{22}} = P_{xx} - P_{yy} + 2iP_{xy}, \quad P^{21} = P^{12} = P_{xx} + P_{yy}, \quad \omega^{11} = \overline{\omega^{22}} = 0, \quad \omega^{21} = \overline{\omega^{12}} = 2i\omega_{xy}. \tag{4}$$

The linear-elasticity formulas differ from formulas (3) in the fact that they do not have nonlinear terms containing the parameter k [5, 6].

To the system of constant contour forces (2) corresponds a zero-order main vector. Indeed, if we denote by α the angle between the normal to the contour and the x axis (Fig. 1), then for the Cartesian components of the normal n_x, n_y and of the stress vector p_x, p_y we have

$$n_x = \cos\alpha = \frac{dy}{ds}, \quad n_y = \sin\alpha = -\frac{dx}{ds}, \qquad p_x = p_n\cos\alpha - p_t\sin\alpha = p\,\frac{dy}{ds} + \tau\,\frac{dx}{ds},$$
$$p_y = p_n\sin\alpha + p_t\cos\alpha = -p\,\frac{dx}{ds} + \tau\,\frac{dy}{ds},$$

where s is the arc of the contour L. It is readily seen that the Cartesian (and also complex) components of the main vector of the contour forces vanish by virtue of the single-valuedness of the contour equations $x = x(s)$ and $y = y(s)$:

$$F_x = \oint_L p_x\,ds = p\oint_L dy + \tau\oint_L dx = 0, \quad F_y = \oint_L p_y\,ds = -p\oint_L dx + \tau\oint_L dy = 0, \quad F = F_x + iF_y = 0.$$

Generally speaking, in a simply connected infinite domain, the complex potentials are nonsingle-valued. The requirement that the stresses and rotation should be single-valued leads to the single-valuedness of the functions $\varphi(z)$ and $\psi'(z)$ [4]. The potentials themselves are representable in terms of the single-valued functions $\varphi_1(z)$ and $\psi_1(z)$ by the formulas

$$\varphi(z) = \varphi_1(z), \quad \psi(z) = B\ln z + \psi_1(z), \quad B = \text{const}. \tag{5}$$

Expanding the single-valued functions into the Laurent series

$$\varphi_1(z) = \sum_{-\infty}^{\infty} A_n z^n, \quad \psi_1(z) = \sum_{-\infty}^{\infty} B_n z^n$$

and using the limitedness of the stresses and rotation at infinity (3), we find that potentials (5) have the form

$$\varphi(z) = A_1 z + \varphi_0(z), \quad \psi(z) = B\ln z + B_1 z + \psi_0(z), \tag{6}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are functions that are holomorphic in the neighborhood of an infinitely distant point; A_1 and B_1 are determined by the conditions at infinity, and B is determined by the main vector of the contour forces:

$$\varphi_0(z) = \sum_{0}^{\infty} A_{-n} z^{-n}, \qquad \psi_0(z) = \sum_{0}^{\infty} B_{-n} z^{-n}, \qquad B = \frac{1}{2\pi}\,\overline{F} = 0, \tag{7}$$

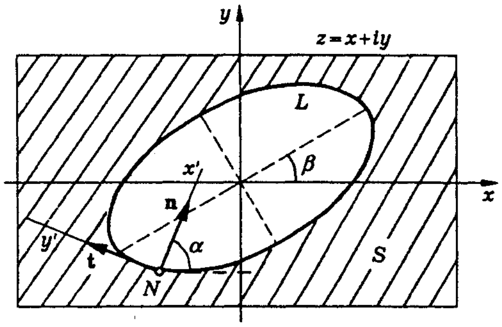

Fig. 1

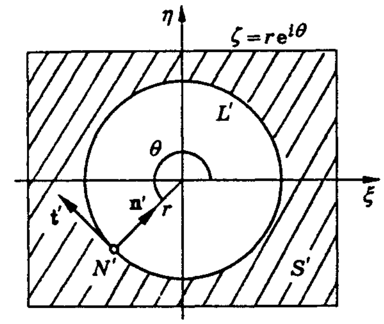

Fig. 2

$$A_1 = \frac{1}{4k}\Big[k(P_{xx}^{\infty} + P_{yy}^{\infty}) + (\omega_{xy}^{\infty})^2 + 2i\omega_{xy}^{\infty}\Big], \quad B_1 = \frac{1}{2}\Big(P_{yy}^{\infty} - P_{xx}^{\infty} + 2iP_{xy}^{\infty}\Big).$$

Let us map conformally the exterior S of the orifice onto the exterior S' of a circle with unit radius (Figs. 1 and 2) with correspondence of infinitely distant points by means of a holomorphic function:

$$z = w(\zeta) = c\zeta + w_0(\zeta), \quad c = \text{const}, \quad w_0(\zeta) = \sum_0^{\infty} c_n\zeta^{-n}, \quad \zeta = re^{i\theta} \in S'. \tag{8}$$

We assume that the constant c is real and positive ($c = \bar{c} > 0$). This corresponds to the absence of rotation of the neighborhood of an infinitely distant point in mapping. Then the complex potentials (6) and their derivatives are written as

$$\begin{gathered}\varphi(z) = \varphi(\zeta) = cA_1\zeta + \sum_0^{\infty} a_n\zeta^{-n}, \qquad \psi(z) = \psi(\zeta) = cB_1\zeta + \sum_0^{\infty} b_n\zeta^{-n}, \\ \Phi(z) = \varphi'(z) = \frac{\varphi'(\zeta)}{w'(\zeta)} = \Phi(\zeta), \quad \Phi'(z) = \varphi''(z) = \frac{\Phi'(\zeta)}{w'(\zeta)}, \quad \Psi(z) = \psi'(z) = \frac{\psi'(\zeta)}{w'(\zeta)} = \Psi(\zeta),\end{gathered} \tag{9}$$

and the complex stresses and rotation (3) take the form

$$\begin{gathered}P^{11} = \overline{P^{22}} = -\frac{2}{\overline{w'(\zeta)}}\Big\{\overline{w'(\zeta)}\,\overline{\Psi(\zeta)} + \overline{\Phi'(\zeta)}\Big[w(\zeta)\big(1 + k\,\overline{\Phi(\zeta)}\big) - k\varphi(\zeta)\Big]\Big\}, \\ P^{21} = 2\Big[\Phi(\zeta) + \overline{\Phi(\zeta)}\Big] + k\Big[\Phi(\zeta) - \overline{\Phi(\zeta)}\Big]^2, \qquad \omega^{21} = 2k\Big[\Phi(\zeta) - \overline{\Phi(\zeta)}\Big].\end{gathered} \tag{10}$$

The normal n and tangent t at point N on contour L can be regarded as the Cartesian x', y' axes rotated about the x, y axes through angle α between the normal and the x axis (see Fig. 1). With allowance for (4), the formulas for the transformation of the complex stress components with rotation of the axis through this angle are written as

$$P_{x'x'} - P_{y'y'} + 2iP_{x'y'} = P'^{11} = P^{11}\,e^{-2i\alpha}, \quad P_{x'x'} + P_{y'y'} = P'^{21} = P^{21}. \tag{11}$$

The magnitude of the angle α is determined by means of transformation of the elementary displacement along the normal to the contour by conformal mapping (Figs. 1 and 2):

$$\begin{gathered}dz = |dz|\,e^{i\alpha}, \quad d\zeta = |d\zeta|\,e^{i(\theta-\pi)} = -\,|d\zeta|\,\frac{\zeta}{|\zeta|}, \\ e^{i\alpha} = \frac{dz}{|dz|} = \frac{w'}{|w'|}\,\frac{d\zeta}{|d\zeta|} = -\frac{w'(\zeta)}{|w'(\zeta)|}\,\frac{\zeta}{|\zeta|}, \quad e^{2i\alpha} = \frac{w'(\zeta)}{\overline{w'(\zeta)}}\,\frac{\zeta}{\bar{\zeta}}.\end{gathered} \tag{12}$$

After this, formulas (11) [with allowance for (12) and the coincidence of the axes $x' = n$ and $y' = t$] have the form

$$P_{nn} - P_{tt} - 2iP_{nt} = \frac{\zeta}{\bar{\zeta}} \frac{w'(\zeta)}{\overline{w'(\zeta)}} \overline{P^{11}}, \quad P_{nn} + P_{tt} = P^{21} \quad \text{on} \quad L. \tag{13}$$

Finally, use of boundary conditions (2) and representations (10) of complex stresses reduces (13) to the following boundary problem for the complex potentials and the mapping function:

$$\sigma - p + 2i\tau = \frac{2\zeta^2}{\overline{w'(\zeta)}} \left\{ w'(\zeta)\Psi(\zeta) + \Phi'(\zeta) \left[\overline{w(\zeta)}(1 + k\,\Phi(\zeta)) - k\overline{\varphi(\zeta)} \right] \right\}, \quad |\zeta| = 1; \tag{14}$$

$$\sigma + p = 2\left[\Phi(\zeta) + \overline{\Phi(\zeta)}\right] + k\left[\Phi(\zeta) - \overline{\Phi(\zeta)}\right]^2, \quad |\zeta| = 1. \tag{15}$$

The expressions of mapping (8) and potentials (9) allow us to establish that the functions considered have the following orders at infinity:

$$\begin{gathered} w(\zeta) = c\zeta + O(\zeta^0), \qquad w'(\zeta) = c + O(\zeta^{-2}), \qquad \Psi(\zeta) = B_1 + O(\zeta^{-2}), \\ \varphi(\zeta) = cA_1\zeta + O(\zeta^0), \qquad \Phi(\zeta) = A_1 + O(\zeta^{-2}), \qquad \Phi'(\zeta) = O(\zeta^{-3}). \end{gathered} \tag{16}$$

The potential $\Phi(\zeta)$, as can be seen from (16), is limited at infinity and is determined by boundary condition (15), which expresses the constancy of the combination of the real and complex parts of the potential on the contour. These conditions are satisfied if the potential is considered constant everywhere in the infinite domain: $\Phi(\zeta) = \text{const}$. Using the corollary of relations (10) $\Phi(\zeta) = (1/(4k))[kP^{21} + \omega^{21}(1 - \omega^{21}/4)]$ and expression (7), we find that this constant is determined by the conditions at infinity:

$$\Phi(\zeta) = \frac{1}{4k}\left[kP^{21}_{\infty} + \omega^{21}_{\infty}\left(1 - \frac{\omega^{21}_{\infty}}{4}\right)\right] = \frac{1}{4k}\left[k(P^{\infty}_{xx} + P^{\infty}_{yy}) + (\omega^{\infty}_{xy})^2 + 2i\omega^{\infty}_{xy}\right] = A_1. \tag{17}$$

If the potential $\Phi(\zeta)$ is known, condition (15) becomes the equation

$$\sigma = P^{\infty}_{xx} + P^{\infty}_{yy} - p. \tag{18}$$

Thus, according to (18), σ is determined by the contour and peripheral loads.

By virtue of (17), $\Phi'(\zeta) = 0$, and hence equality (14) becomes the boundary condition for finding the functions $w(\zeta)$ and $\Psi(\zeta)$:

$$E\,\overline{w'(\zeta)} = \zeta^2 w'(\zeta)\Psi(\zeta), \quad |\zeta| = 1, \quad E = E_1 + iE_2 = (1/2)(P^{\infty}_{xx} + P^{\infty}_{yy}) - p + i\tau. \tag{19}$$

Using analytical continuation, we represent (19) in the form of a functional equation which is true on the exterior of the unit circle,

$$E\,\overline{w}'(1/\zeta) = \zeta^2 w'(\zeta)\Psi(\zeta), \quad \zeta \in S' \tag{20}$$

and use it to determine the desired functions.

We shall seek the constituent $w_0(\zeta)$ of mapping (8) in the form of a polynomial in odd powers of the argument, so that the mapping itself takes the form

$$w_0(\zeta) = \sum_{0}^{l} c_{2k+1}\zeta^{-(2k+1)}, \quad w(\zeta) = c\zeta + c_1\zeta^{-1} + c_3\zeta^{-3} + \ldots + c_{2l+1}\zeta^{-(2l+1)}, \quad c = \bar{c}. \tag{21}$$

The polynomial degree is found from the condition that the orders of the left- and right-hand sides of Eq. (20) coincide at infinity.

We establish on the basis of (21) that the derivatives of the mapping entering into (20) have the following orders at infinity:

$$\begin{gathered} w'(\zeta) = c - c_1\zeta^{-2} - \ldots - (2l+1)c_{2l+1}\zeta^{-(2l+2)} = O(\zeta^0), \\ \bar{w}'(1/\zeta) = c - \bar{c}_1\zeta^2 - \ldots - (2l+1)\bar{c}_{2l+1}\zeta^{2l+2} = O(\zeta^{2l+2}). \end{gathered}$$

From here and from (16), it can be seen that, according to Eq. (20), $O(\zeta^{2l+2}) = O(\zeta^2)$ for $\zeta \to \infty$. Consequently, the orders of the sides of the equation agree for $l = 0$. Thus, the mapping function (21) contains two parameters (real and complex) and has the form

$$z = w(\zeta) = c\zeta + c_1/\zeta = n(\zeta + m/\zeta), \qquad n = \bar{n} = c, \qquad m = m_1 + im_2 = c_1/c. \tag{22}$$

Using (20) and (22), we find another potential

$$\Psi(\zeta) = E(1 - \overline{m}\zeta^2)/(\zeta^2 - m), \tag{23}$$

where the constant E is given by formula (19).

To potentials (17) and (23) correspond the following values of the complex components of stresses and rotation (10):

$$P^{11} = \overline{P^{22}} = 2\overline{E}\,\frac{m\bar{\zeta}^2 - 1}{\bar{\zeta}^2 - \overline{m}}, \qquad P^{21} = P^{\infty}_{xx} + P^{\infty}_{yy}, \qquad \omega^{21} = 2i\omega^{\infty}_{xy}. \tag{24}$$

Letting $\bar{\zeta} \to \infty$ in the first of these, we obtain the equality $P^{11}_{\infty} = 2\overline{E}m$, which, with allowance for conditions (1) and (19), determines the parameter m:

$$m = m_1 + im_2 = \frac{P^{\infty}_{xx} - P^{\infty}_{yy} + 2iP^{\infty}_{xy}}{P^{\infty}_{xx} + P^{\infty}_{yy} - 2p - 2i\tau}. \tag{25}$$

Here

$$m_1 = \frac{(P^{\infty}_{xx} - P^{\infty}_{yy})(P^{\infty}_{xx} + P^{\infty}_{yy} - 2p) - 4\tau P^{\infty}_{xy}}{(P^{\infty}_{xx} + P^{\infty}_{yy} - 2p)^2 + 4\tau^2}, \quad m_2 = 2\,\frac{P^{\infty}_{xy}(P^{\infty}_{xx} + P^{\infty}_{yy} - 2p) + \tau(P^{\infty}_{xx} - P^{\infty}_{yy})}{(P^{\infty}_{xx} + P^{\infty}_{yy} - 2p)^2 + 4\tau^2}. \tag{26}$$

Thus, the parameter m appearing in (22) is determined by the given contours and peripheral loads.

In accord with the choice of a mapping, the full-strength contour will be the curve into which the circumference of the unit circle is mapped in the case of the mapping function (22). Assuming that $\zeta = \exp(i\theta)$ for points on this circumference and separating the real and complex parts in (22), we obtain parametric equations for the full-strength contour:

$$x = n[(1 + m_1)\cos\theta + m_2 \sin\theta], \quad y = n[m_2\cos\theta + (1 - m_1)\sin\theta]$$

or

$$m_2 x - (1 + m_1)y = -n(1 - m_1^2 - m_2^2)\sin\theta, \quad (1 - m_1)x - m_2 y = n(1 - m_1^2 - m_2^2)\cos\theta. \tag{27}$$

Squaring each of equalities (27) and adding up the results, we exclude the variable parameter θ and obtain an explicit equation of a full-strength contour in the form of a central two-order curve:

$$\begin{gathered} a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0, \quad a_{11} = (1 - m_1)^2 + m_2^2, \quad a_{22} = (1 + m_1)^2 + m_2^2, \\ a_{33} = -n^2(1 - m_1^2 - m_2^2)^2, \quad a_{12} = a_{21} = -2m_2, \quad a_{13} = a_{31} = 0, \quad a_{23} = a_{32} = 0. \end{gathered} \tag{28}$$

The invariants I, D, and A of this curve, and also the quantity A' determined by the curve, have the form

$$\begin{gathered} I = a_{11} + a_{22} = 2(1 + m_1^2 + m_2^2), \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (1 - m_1^2 - m_2^2)^2, \\ A = \det(a_{kl})^3_{k,l=1} = -n^2(1 - m_1^2 - m_2^2)^4, \\ A' = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = -2n^2(1 + m_1^2 + m_2^2)(1 - m_1^2 - m_2^2)^2. \end{gathered} \tag{29}$$

In the general case, the contour and peripheral loads and, hence, parameters (26) are independent of one another, and this indicates the validity of the inequality

$$1 - m_1^2 - m_2^2 \neq 0, \tag{30}$$

which is equivalent, by virtue of (25), to the inequality $(P^{\infty}_{xx} - P^{\infty}_{yy})^2 + 4(P^{\infty}_{xy})^2 \neq (P^{\infty}_{xx} + P^{\infty}_{yy} - 2p)^2 + 4\tau^2$.

Then, according to (29), $D > 0$ and $A/I < 0$ and, according to the known criterion [7], Eq. (28) of a full-strength contour is the equation of an ellipse. The center of the ellipse coincides with the Cartesian coordinate origin. The angle β between the positive direction of the abscissa and the axes of symmetry of the ellipse is defined by the relation

$$\tan 2\beta = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{m_2}{m_1}, \tag{31}$$

and its semi-axes a and b are expressed in terms of the invariants and roots λ_1 and λ_2 ($\lambda_1 \geqslant \lambda_2$) of the characteristic equation $\lambda^2 - I\lambda + D = 0$ and in terms of parameters (26) by the formulas

$$a^2 = -\lambda_1 \frac{A}{D^2} = \frac{n^2}{2}\left(I + \sqrt{I^2 - 4D}\,\right) = n^2\left(1 + \sqrt{m_1^2 + m_2^2}\,\right)^2,$$

$$b^2 = -\lambda_2 \frac{A}{D^2} = \frac{n^2}{2}\left(I - \sqrt{I^2 - 4D}\,\right) = n^2\left(1 - \sqrt{m_1^2 + m_2^2}\,\right)^2.$$

Hence,

$$a_\pm = n\left(1 + \sqrt{m_1^2 + m_2^2}\,\right), \quad b_\pm = \pm n\left(1 - \sqrt{m_1^2 + m_2^2}\,\right), \tag{32}$$

where the upper sign appears for $\sqrt{m_1^2 + m_2^2} < 1$ and the lower sign for $\sqrt{m_1^2 + m_2^2} > 1$. In turn, the parameters of the ellipse are expressed in terms of the semi-axes as

$$n = \frac{1}{2}(a_\pm \pm b_\pm), \qquad \sqrt{m_1^2 + m_2^2} = \frac{a_\pm \mp b_\pm}{a_\pm \pm b_\pm},$$

whence it can be seen that the parameter m in (22) characterizes the shape of the ellipse, and the parameter n its dimensions; the former is determined by the load applied to the plate, and the latter remains arbitrary. The fact that the semi-axes of ellipse (32) are proportional to the arbitrary parameter, and the slope of the axes of symmetry (31) is independent of it means that full-strength contours form a one-parameter set of similar ellipses. This result is similar to the conclusion obtained in the linear theory of elasticity [1] by a different method.

When invariants (29) and parameters (26) are related to one another by

$$0 = I^2 - 4D = 16(m_1^2 + m_2^2) \quad (m_1 = m_2 = 0),$$

the semi-axes (32) of the ellipse coincide, and the directions of its axes of symmetry (31) become indefinite: the ellipse degenerates into a circle. To this case, in view of (26), correspond the following relations between the load elements:

$$(P_{xx}^\infty - P_{yy}^\infty)^2 + 4(P_{xy}^\infty)^2 = 0, \quad (P_{xx}^\infty + P_{yy}^\infty - 2p)^2 + 4\tau^2 \neq 0.$$

Specifically, these relations are realized in the case of "overall" extension at infinity ($P_{xx}^\infty = P_{yy}^\infty = P_0$ and $P_{xy}^\infty = 0$) and arbitrary contour loads [excluding $(P_0 - p)^2 + \tau^2 = 0$].

An alternative situation arises when, in place of inequality (30), we have the equality

$$1 - m_1^2 - m_2^2 = 0, \tag{33}$$

which, by virtue of (26), is equivalent to the following condition imposed on the load:

$$(P_{xx}^\infty - P_{yy}^\infty)^2 + 4(P_{xy}^\infty)^2 = (P_{xx}^\infty + P_{yy}^\infty - 2p)^2 + 4\tau^2. \tag{34}$$

Then $D = 0$ and $A' = 0$. According to the known criteria [7], these conditions imply that Eqs. (27) of the full-strength contour give a second-order degenerate quadratic curve, i.e., a pair of coinciding lines which pass through the origin, $m_2 x - (1 + m_1)y = 0$ and $(1 - m_1)x - m_2 y = 0$, whose slope to the x axis is given by the formula $\tan\beta = (1 - m_1)/m_2 = m_2/(1 + m_1)$ resulting from (31) and (33).

The orifice itself is defined as a segment of these lines. Indeed, the case of (33) can be regarded as the limiting value of general condition (30) when $m_1^2 + m_2^2 \to 1$. In this passage to the limit, semi-axes (32) of the ellipse take the form $a_\pm = a = 2n$ and $b_\pm = b = 0$, and this indicates that the ellipse degenerates into a rectilinear slot of length $4n$ directed at an angle β to the x axis. Since the length of the slot, unlike the slope, depends on the free parameter, in this case, too, the full-strength contours constitute a one-parameter set of slots belonging to the same line and having the same center of symmetry.

In particular, condition (34) is realized for the "overall" tension of intensity P_0 ($P_{xx}^{\infty} = P_{yy}^{\infty} = P_0$ and $P_{xy}^{\infty} = 0$) at infinity and under a normal contour load of the same intensity ($p = P_0$ and $\tau = 0$).

Thus, under a load of the general form, the full-strength contours will be ellipses. But if the load is subject to certain conditions, the ellipses degenerate into circles or rectilinear slots.

In conformal mapping (22), to the polar r, θ coordinates in the plane of a unit circle correspond elliptical coordinates in the plane of the plate. The physical components of stresses and rotation in these coordinates P_{rr}, $P_{\theta\theta}$, $P_{r\theta}$, and $\omega_{r\theta}$ are related to the complex components by [8]

$$P_{rr} - P_{\theta\theta} + 2iP_{r\theta} = \frac{\bar{\zeta}}{\zeta}\frac{\overline{w'(\zeta)}}{w'(\zeta)} P^{11}, \quad P_{rr} + P_{\theta\theta} = P^{21}, \quad 2i\omega_{r\theta} = \omega^{21}.$$

Substitution of complex components (24) and of mapping (22) into these relations leads to the following stress and rotation fields in the elliptical coordinates:

$$\begin{aligned} P_{rr} &= G + \frac{E_1}{e}\left[(r^4+1)f(\theta) - r^2(1+m_1^2+m_2^2)\right] + \frac{E_2}{e}(r^4-1)g(\theta), \\ P_{\theta\theta} &= G - \frac{E_1}{e}\left[(r^4+1)f(\theta) - r^2(1+m_1^2+m_2^2)\right] - \frac{E_2}{e}(r^4-1)g(\theta), \\ P_{r\theta} &= \frac{E_1}{e}(r^4-1)g(\theta) - \frac{E_2}{e}\left[(r^4+1)f(\theta) - r^2(1+m_1^2+m_2^2)\right], \quad \omega_{r\theta} = \omega_{xy}^{\infty}, \end{aligned} \tag{35}$$

where $G = (1/2)(P_{xx}^{\infty} + P_{yy}^{\infty})$; $e = r^4 - 2r^2 f(\theta) + m_1^2 + m_2^2$; $f(\theta) = m_1 \cos 2\theta + m_2 \sin 2\theta$; $g(\theta) = m_2 \cos 2\theta - m_1 \sin 2\theta$; and m_1, m_2, and E_1, E_2 are determined by expressions (26) and (19). Formulas (28) and (35) solve the problem of a full-strength orifice.

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